Mass Splittings From Symmetry Obstruction

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ABSTRACT

The long-range fields associated with non-abelian vortices generally obstruct full realization, in the spectrum, of the symmetry of the ground state. In the context of 2+1 dimensional field theories, we show how this effect manifests itself concretely in altered conditions for the angular momentum and in the energy spectrum. A particularly interesting case is supersymmetry, which is obstructed by the gravitational effect of any mass.

The existence of fundamental symmetries that are somehow hidden – spontaneously broken, or asymptotic – is an important theme in modern physical theory. There is another mechanism for hiding symmetry that has been less analyzed, and does not seem to have acquired a name, though it is known to occur in various contexts. This phenomenon, which we propose to call obstructed symmetry, is connected with the existence of long-range gauge fields (or potentials). Symmetry transformations among charged fields are implemented, in quantum gauge field theory, by operators involving an integral over a surface at spatial infinity of the electric field. In a non-abelian theory the electric field itself is gauge covariant, not invariant, so that to define the surface integral uniquely one must have a consistent global definition of the gauge frames over the surface. Long-range gauge fields, even if they reduce to locally trivial gauge potentials, can obstruct such a definition. In that case, the usual consequences of the symmetry may not hold [1, 2, 3].

All this is rather abstract, and our first goal in this note is just to exemplify (conceptually) tangible consequences of obstructed symmetry in the simplest possible context, that of charges bound to non-abelian vortices in 2+1 dimensions. We show explicitly how the angular momentum quantization conditions are modified, depending on internal quantum numbers, in such a way that the symmetry of the energy spectrum is reduced. Essentially the same phenomenon occurs universally in 2+1 dimensional supergravity, and plays an important role in some recent ideas concerning the vanishing of the cosmological constant [3, 4, 5]. It is important for these ideas that supersymmetry is not broken, but that its obstruction leads to fermion-boson splitting. We show explicitly, in a special case, how this occurs.

Non-abelian vortices:

A nonabelian vortex is characterized in regular gauge by a non-zero matrix vector potential a_θ at spatial infinity, and the holonomy

$$h = \mathcal{P}e^{i\int_0^{2\pi} a_\theta \, d\theta} \, .\tag{1}$$

Here $\mathcal{P}$ denotes path ordering, and a is a matrix in the adjoint representation. (More precisely, h is defined only up to conjugacy, since a gauge transformation $\Lambda(r,\theta) \to \Lambda_\infty(\theta)$ takes $h \to \Lambda_\infty(2\pi)h\Lambda_\infty(0)^{-1}$.) We may suppose, after a non-singular gauge transformation, that a_θ is a constant matrix. Given a charged field η, one has its covariant derivative

$$D_\mu\eta \;=\; \partial_\mu\eta + iT^a a_\mu^{\mathrm{a}}\eta \;. \tag{2}$$

Here T^{a} are the generators for the representation to which η belongs, and of course the a^{a} are the internal components of a. The physical implication of the vector potential emerges clearly if one diagonalizes $T^{\mathrm{a}}a_\mu^{\mathrm{a}}$ (assumed constant) and considers the partial wave $\eta(r,\theta) = e^{il\theta}f(r)$. In the regular gauge l is quantized to be an integer, but for the kth eigenvalue $e_{(k)}$ and eigenvector $\eta_{(k)}$ one has the azimuthal derivative

$$D_\theta\eta_{(k)} \;=\; i(l + e_{(k)})\eta_{(k)} \;. \tag{3}$$

Thus the angular energy term, which has direct physical significance, is proportional to $\frac{(l+e_{(k)})^2}{r^2}$. It behaves as if there is a fractional contribution $e_{(k)}$ to the angular momentum.

Alternatively, by a singular gauge transformation (with gauge function proportional to θ, and thus having a cut) one could formally remove a_θ, at the cost of introducing boundary conditions of the form

$$\eta(2\pi) \;=\; \rho(h)\eta(0) \tag{4}$$

on the azimuthal dependence of the field η (and thus on the wave functions of its quanta). Here of course ρ is the appropriate unitary representation matrix. Upon diagonalizing ρ one finds conditions $\eta_{(k)}(2\pi) = e^{i\phi_{(k)}}\eta_{(k)}(0)$ on the wave functions, in evident notation. In this formulation the azimuthal covariant derivative is simply

the ordinary spatial azimuthal derivative, but one has the condition

$$l_{(k)} = \text{integer} + \frac{\phi_{(k)}}{2\pi} \tag{5}$$

on the partial waves $\eta_{(k)} \propto e^{il\theta} f(r)$. The form of the angular energy term, of course, is the same as before, with $e_{(k)} \equiv \phi_{(k)}/2\pi$.

Generically states with a given value of $e_{(k)}$ (and the same spatial wave functions) will be degenerate, but states with different values of $e_{(k)}$ will not be degenerate, because they see a different effective Hamiltonian – or, alternatively, because they obey a different angular momentum quantization condition. Now suppose that the symmetry of the ground state is a non-abelian group K, and consider $\kappa \in K$. If ρ is a faithful representation, and $h\kappa \neq \kappa h$, then $\rho(\kappa)$ will not act within the spaces of fixed $e_{(k)}$, but will connect states with different values of $e_{(k)}$. As we have seen, generically this means that $\rho(\kappa)$ will connect states of different energy. Thus the obstructed symmetry K will not be realized as a symmetry of the spectrum, but will connect states with different energy. Only members of the centralizer group $\mathcal{C}(h)$ of transformations which commute with h will generate spectral symmetries.

As a simple example, consider the Alice string [6]. It arises from the symmetry breaking $SO(3) \rightarrow SO(2) \times Z_2$ (semidirect product) induced by the vacuum expectation value $T_{11} = T_{22} = \frac{-1}{2} T_{33} \neq 0$ of a traceless symmetric tensor. The residual symmetries are products of rotations in the 1-2 plane and the parity transformation diagonal $(1, -1, -1)$. Ordinarily the representations associated with non-vanishing $SO(2)$ charge Q would fall into two-dimensional multiplets of $SO(2) \times Z_2$ composed of degenerate states with charges Q with $-Q$. However, in the presence of a vortex with holonomy diagonal $(1, -1, -1)$ this doublet breaks up into a symmetric combination with integer effective angular momentum and an antisymmetric combination with effectively half-odd integer angular momentum. In a generic potential, or specifically for example a harmonic oscillator potential, these states are not degenerate.

The Alice string model has the annoying complication that the charge induces a Coulomb field with logarithmically divergent energy, so that to set up the problem properly one must consider neutral configurations, such as particle-antiparticle pairs (and then one gets involved with the peculiarities of Cheshire charge). These problems can be avoided in a different symmetry breaking scheme, where the symmetry of the ground state is a nonabelian discrete group. Perhaps the simplest example is the breaking $SU(2) \rightarrow$ unitquaternions which can be obtained with a symmetric four-index spinor having as its only non-vanishing expectation values $S_{1111} = S_{2222} \neq 0$. The doublet representation of the unit quaternions, in the presence of a vortex with holonomy σ_3, for example, breaks up into two singlets with integer and half-odd integer effective angular momentum, much as in the Alice case.

The effects of obstructed symmetry appear directly only in states carrying both flux and charge. In the preceding discussion, these ingredients were rather artlessly brought together by considering bound states of pure flux and pure charge. A more intrinsic form of the phenomenon occurs in Chern-Simons theories, where the interaction itself naturally associates flux with charge [7]. The question then arises whether splittings are induced for the fundamental quanta, which carry both charge and flux. We certainly expect they will, for the following reason. In evaluating the self-energy of such a quantum, one must sum over space-time trajectories where its world line is self-linked or knotted. The charge from one part of the trajectory gets entangled with the flux emanating from another part, and the propagation is altered, as above. For non-dynamical sources (Wilson lines) it is known that the amplitude of a trajectory depends on its topology; indeed, this fact is at the root of the remarkable application of non-abelian Chern-Simons theories to topology [8]. It would be very interesting to do a dynamical calculation along these lines.

Supersymmetry:

In 2+1 dimensional gravity [9] the primary effect of a point mass is to create

a conical geometry in its exterior, with a deficit angle δ proportional to the mass; $\delta = 2\pi G m$. The exterior geometry is then locally flat, but non-trivial globally. Indeed, it induces a modification in the angular momentum quantization for particles in the exterior, similar in some ways to the non-abelian vortex. A notable difference is that whereas the non-abelian vortex induced an additive change in the quantization condition, change induced by the gravitational field of a point mass is multiplicative. Indeed, from requiring that the azimuthal factor $e^{il\theta}$ be single-valued as $\theta \to \theta + (2\pi - \delta)$ we find $l = \text{integer}/(1 - \frac{\delta}{2\pi})$. This difference reflects the different symmetry of the sources: an additive shift in angular momentum requires an intrinsic orientation violating the discrete symmetries P and T, while a multiplicative factor is even under these transformations.

The conical geometry presents a global obstruction to defining a spinor supercharge (see below), so that supersymmetry, even if valid for the ground state, is obstructed for massive states. One could therefore reasonably expect that fermion-boson degeneracy is lifted [3, 4, 5], but a concrete calculation might be welcome.

Perhaps the simplest way to illuminate the central issue is to consider what is the condition analogous to (5) in this context. One can remove the effect of the conical geometry by a singular gauge transformation, in favor of a modified boundary condition. Here the covariant derivative should respect parallel transport, so that we should require that the effect of orbiting the apex of the cone is to rotate spinors (or vectors) through the appropriate angle, *i.e.* to multiply them by a phase proportional to the spin and angle. An important subtlety, which arises even in flat space, is that one must include a factor -1 for half-odd integer (fermion) fields. This factor is essentially equivalent to the -1 accompanying fermion loops in Feynman graphs, and its inclusion implements the normal spin-statistics relation. With this factor $e^{2\pi i s}$ taken out, the appropriate condition on wave functions $\psi_s \propto f(r)e^{il\theta}$ for spin s quanta, in order that they transform properly, is

$$\psi_s(2\pi - \delta) = e^{i(2\pi - \delta)l - i\delta s}\psi_s(0) \tag{6}$$

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leading to

$$l \;=\; \frac{1}{1-\frac{\delta}{2\pi}}\text{integer} \;+\; s\frac{\frac{\delta}{2\pi}}{1-\frac{\delta}{2\pi}} \;. \tag{7}$$

For our purposes, the crucial aspect of (7) is simply that the quantization condition on l becomes, for $\delta \neq 0$, s-dependent. This certainly suggests a general mechanism for splitting states formally (that is, here, locally) related by supersymmetry. By forbidding $l = 0$ for $s = 1/2$, it also provides the primary obstruction to the construction of well-defined supercharge generators.

To see this mechanism at work concretely, let us consider an analogue of the hydrogen atom. One can construct a supersymmetric version of quantum electrodynamics, containing very massive protons and sprotons and lighter electrons and selectrons, and consider the analogue of the hydrogen atom [10]. The main novelty introduced by supersymmetry is that there is a boson-boson (sproton-selectron) system which mixes with the fermion-fermion system by photino exchange, and similarly a boson-fermion/fermion-boson complex. These two complexes are related by supersymmetry, and form degenerate multiplets.

This superelectrodynamics can be coupled to supergravity [11], and one can consider gravitational corrections to the 'hydrogen' spectrum. One can, in this context, consider the corrections to the effective non-relativistic hamiltonian due to one photino exchange. The computation [10, 11] follows closely the classical derivation of the Breit hamiltonian for positronium and takes the form

$$\Delta\mathcal{H} \;=\; \frac{-i}{2m}\big[P_i, V(x)\big]\Sigma_i \;, \tag{8}$$

where m is the electron mass, V is the Coulomb potential $\frac{e^2}{2\pi}\ln r$, and the Σ_i are matrices acting in the internal spin space of the electron-proton system. (8) does not yet include any coupling to gravity. We expect the form of this correction to the hamiltonian to be unaltered by gravitational effects, since $\Delta\mathcal{H}$ is a local operator on the wavefunction and gravity in 2+1 dimensions has only global topological effects.

This expectation is reinforced by its consistency with the boundary conditions mentioned above. Indeed the exact form of the matrices Σ_i implies that, in a conical geometry, the operator $\Delta\mathcal{H}$ becomes hermitian (in a non-trivial way) if the boundary condition (6)is satisfied by the components of the wavefunction. For the spin matrices connect components of the wavefunction with internal spin differing by 1, and the phase acquired by the operator $[P_i, V(x)]$ (or simply by x_i after taking the commutator), which would otherwise spoil hermiticity, is exactly compensated by the different, spin-dependent, boundary conditions on these components.

The main effect of the gravitational field, however, arises already in the nominally spin-independent interaction, which dominates in the low-velocity (non-*special* relativistic) limit: the modified condition (7) for the allowed angular momenta modifies the effective Schrodinger equation, and splits the spectrum corresponding to different spin values. Here we are concerned with the motion of the electron in the geometry and potential provided by the proton, and take $\delta = 2\pi G M$. Clearly supersymmetry is obstructed, and it is not manifest in the spectrum.

We have glossed over several technicalities that do not substantially affect this leading-order calculation, though in extending it to higher orders in e^2 and m/M they would need to be carefully addressed. The most interesting, perhaps, comes from the double-conical geometry arising when one includes the gravitational fields of both particles. One then obtains a modified quantization condition on the relative angular momentum, which takes the form

$$l_{\text{rel.}} = \frac{\text{integer}}{1 - \frac{\delta_{\max}}{2\pi}} + \frac{1}{2\pi}\frac{(\delta_1 s_2 + \delta_2 s_1)}{1 - \frac{\delta_{\max}}{2\pi}} \,, \tag{9}$$

where δ_i, s_i are the deficit angle and spin associated with particle i, and $\delta_{\max}$ is the larger deficit angle. Also one should use the Kerr geometry to take into account the spin and angular momentum of the particles, but the effects on the quantization condition of doing so are subleading, of relative order $(s$ or $l)\delta$.

Extension of the calculation to higher orders in G raises questions on another level, since in coupling supergravity to matter we are inevitably dealing with non-renormalizable theories, at least perturbatively. The foregoing considerations, which depend only on the behavior of the theory at low energies, should presumably hold good independent of how the theory is ultimately cut off at very high energies. The non-renormalizability would seem to present a serious barrier to calculating obstructed supersymmetry splittings due to self-energy, however. We certainly expect such corrections to arise from Feynman graphs with non-trivial topological structure, for reasons similar to those mentioned above in connection with nonabelian Chern-Simons theories. Indeed, one can formulate the pure gravity theory, at least, as a special sort of non-compact nonabelian Chern-Simons theory [12] (which is not only renormalizable, but topological). Perhaps for some strongly coupled supersymmetric theory the ultraviolet behavior is sufficiently improved that the theory remains renormalizable, non-perturbatively, even after coupling to supergravity.

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